P''', respectively, giving us, therefore, two inscribed triangles, PQR and $P''Q''R''$, and two circumscribed triangles $P'Q'R'$ and $P'''Q'''R'''$. We will now show that O is the common co-polar point of all four triangles, and HJI, along its polar, their three common collinear co-axial points.

For, by construction, O is the co-polar point of the inscribed triangles PQR and $P''Q''R''$. And if their sides PR and $P''R''$ cut in, say h, while QR and $Q''R''$ cut in, say i, and PQ, $P''Q''$ cut in, say j; then will O and h be summits of quadrangle $PRR''P''$, and likewise Oi summits of quadrangle $QRR''Q''$, and Oj summits of quadrangle $PQQ''P''$. Hence hji is the polar of O with respect to the conic; thus coinciding with line HJI. But the three points where the polar of O, the co-axial line of triangles PQR and $P'Q'R$, cuts the sides of triangle PQR, are HJI. So that lines HJI and hji not only coincide; but points Hh, Jj, and Ii, respectively, are identical; so that HJI are the common collinear co-axial points of the three co-polar triangles PQR, $P'Q'R'$, and $P''Q''R''$; their corresponding sides PR, $P'R'$, and $P''R''$ concurring in H; sides PQ, $P'Q'$, and $P''Q''$ concuring in J, and QR, $Q'R'$, and $Q''R''$ in I.

Then lastly, H being a summit of the quadrangle $PJIR$, must be the pole to the polar $QQ'Q''O$ joining its other two summits Q' and Q, and hence in it meet the two tangents $QP'HR'$ and $R'''Q''P'''H$, while the quadrangle $PJIR$ again gives us $RQ'JP'$ as a harmonic range. And thus $PQ'I$ and $OPP'P''$ as conjugate rays to the pencil $PQ'I$, PJQ, $PP'P''$, and RPH. So that in I concur the tangents $R'PQ'I$ and $P''Q'''IR'''$. While similarly, $RQ'JP'$ and $RR''R'$ being conjugate rays in the pencil RQ, RP', RH, and RR', it follows that in J concur the tangents $RQ'J$, and $R''P'''JQ'''$.

So that HJI are the three collinear points in which the two inscribed triangles PQR and $P''Q''R''$, and the two circumscribed triangles $P'Q'R'$ and $P'''Q'''R'''$ are commonly co-axial, while by Theorem 1, triangles $P''Q''R''$ and $P'''Q'''R'''$, being so co-axial in HJI, must be co-polar in O, its pole, and thus all four triangles be commonly co-polar in said point O. So that $OPP'''P'P''$ are collinear; as also $OQ''Q'Q'''Q$, and $R'''ROR''R'$.

NOTE ON THE QUARTIC.

By DR. R. P. STEPHENS, Wesleyan University, Middletown, Conn.

The general quartic equation

$$ax^4+4bx^3+6cx^2+4dx+e=0, \tag{1}$$

where the coefficients may be real or complex, can be reduced to the form

$$x^4-2\mu x^2+\nu^2=0. \tag{2}$$

The roots of (2) are

$$x = \pm\sqrt{[\mu \pm \sqrt{(\mu^2 - \nu^2)}]} \text{ or } x = \pm\frac{1}{\sqrt{2}}[\sqrt{(\mu-\nu)} \pm \sqrt{(\mu+\nu)}].$$

If we make the substitution

$$\sqrt{2}\, t \equiv \sqrt{(\mu-\nu)} \pm (\mu+\nu),$$

then the four roots may be expressed as

$$x_1 = t, \quad x_2 = -t, \quad x_3 = \nu/t, \quad x_4 = -\nu/t.$$

That any quartic which has no repeated roots can be reduced to the form (2) is easily seen geometrically. In general, the roots of a quartic are represented by four points in the complex plane, say x_1, x_2, x_3, x_4. If these be divided into pairs x_1, x_2 and x_3, x_4, there is a pair α, β which is harmonic* with respect to both. That bilinear transformation which throws α into zero and β into infinity will obviously throw x_1 and x_2 into points symmetrical with respect to the origin, and will arrange x_3 and x_4 in a similar manner; and the transformed points will be as given above.

The reduction of the quartic to this simple form is secured analytically as follows: If x_1, x_2, x_3, x_4 are the four roots of (1), then the pair α, β will be harmonic with respect to the two pairs x_1, x_2 and x_3, x_4, provided

$$2\alpha\beta - (\alpha+\beta)(x_1+x_2) + 2x_1x_2 = 0,$$
$$\text{and } 2\alpha\beta - (\alpha+\beta)(x_3+x_4) + 2x_3x_4 = 0;$$

from which
$$\alpha + \beta = \frac{2(x_1x_2 - x_3x_4)}{x_1x_2(x_3+x_4) - x_3x_4(x_1+x_2)}.$$

But since the four roots may be grouped in two other ways, we shall have two other values for $(\alpha+\beta)$. Forming the equation of which these three values of $\alpha+\beta$ are roots and substituting the values for symmetric functions of the roots from (1), we obtain†

(3)
$$2(a^2d - 3abc + 2b^3)\lambda^3 + (a^2e + 6b^2c - 9ac^2 + 2abd)\lambda^2$$
$$+ 2(abc + 2b^2d - 3acd)\lambda + b^2e - ad^2 = 0,$$

where $\lambda \equiv \alpha + \beta$.

In a similar way, from the reciprocals of the roots, we obtain

(4)
$$2(be^2 - 3cde + 2d^3)\lambda'^3 + (ae^2 + 6cd^2 - 9c^2e + 2bde)\lambda'^2$$
$$+ 2(ade + 2bd^2 - 2bce)\lambda' + ad^2 - b^2e = 0,$$

*For this general use of harmonic pairs, see Harkness and Morley, *Introduction to Analytic Functions*, pp. 32—34.

†See Burnside and Panton, *Theory of Equations*, p. 130 (3rd Ed.).

where $\lambda' \equiv 1/\alpha + 1/\beta$. From λ and λ', we can obtain α and β.

If now α_1, β_1 are a harmonic pair obtained from (3) and (4), then the bilinear transformation

$$x = \frac{\beta_1 x' + \alpha_1}{x' + 1}$$

will reduce the general quartic (1) to the required form.

In case two of the roots of (1) are equal, say $x_1 = x_2$, then equation (3) will have a repeated root which will equal twice the reciprocal of x_1, the repeated root of (1); and if three roots of (1) are equal, then all harmonic pairs coincide at the repeated root and (3) will be a perfect cube. Thus we see that the general quartic with real or complex coefficients can be reduced to the form (2) in this way.

REMARKS ON THE REPORT ON GEOMETRY OF THE COMMITTEE* OF THE CENTRAL ASSOCIATION.

By PROFESSOR GEORGE BRUCE HALSTED, Greeley, Colorado.

This report is epoch-making,—at the very least epoch-marking.

There simply *must* be things not *explicitly* defined, and point, straight, plane, between, should be among them. The Report says: "It is recommended that the term *sect* be used for segment of a straight line lying between two of its points." In fact the term "sect" has 'arrived.' The Encyclopaedia Americana uses it. This Report uses it more than twenty-two times. It is used not less than twenty-four times in the remarkable Presidential Address by Professor Alfred Baker of the University of Toronto to the Royal Society of Canada.

Think of the neatness with which the bunglesome phrase "transferrer of straight-line segments" becomes *sect-carrier*. Realize how elegantly "the algebra or algorithm of straight-line segments" becomes *sect-calculus*.

"Instead of *axioms*," says the Report, "use geometrical *assumptions*." It gives for example Pasch's assumption, now so renowned. From the list of assumptions we mention: "3. A point on a straight line divides it into two parts, called *rays*." "6. A sect has one and only one mid point." "11. A straight line divides the points not on it into two classes such that sects determined by two points of the same class are not intersected by the line, and sects determined by two points not of the same class are intersected by the line."

*The Committee consist of G. W. Greenwood, Chairman, Salem, Va.; C. A. Pettersen, Chicago, Ill.; C. E. Comstock, Peoria, Ill., and C. W. Newhall; Faribault, Minn. Copies of the report may be had by sending a stamp to Miss Mabel Syker, 438 East 57th Street, Chicago, Ill. ED. S.